Structural features of external representations and implications for physics instruction

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To better understand how external representations support learning and doing physics, we present a theoretical analysis of representations informed by perspectives from cognitive science, including information processing and distributed cognition. We describe external representations in terms of three aspects: organization of information about a conceptual referent presented in a medium. To provide a more detailed description that is more closely linked to instruction and learning, we consider the intersection of the form and function of representations and identify a set of nine structural features. For example, literalness refers to how literally the representation captures the thing being represented. We illustrate these structural features with four examples where we compare and contrast the characteristics of different representations. We argue that learners and professionals often benefit from different structural features and discuss ways of sequencing representations for learners.

I. INTRODUCTION

External representations are representations that exist ‘outside the head’ such as a diagrams on a whiteboard, algebra written on a napkin, or a physical model of a crystal unit cell. Physicists use external representations to solve problems, calculate, structure their reasoning, and communicate their ideas about the physical world. The process of becoming a physicist includes learning to interpret and use a standard set of specialized representations (e.g., free-body diagrams, phase diagrams, circuit diagrams, bandgap diagrams, Dirac bra-ket notation, and Feynman diagrams). Therefore, helping students develop fluency with external representations is an important goal for physics educators and a topic of physics education research [1–3]. Specifically, students can learn to interpret and use external representations to understand physics content (and external representations may be considered physics content themselves), solve problems, and engage in important practices of the discipline. In particular, physics learners are trained to use representations to analyze physical systems in particular ways, and to quickly interpret particular features of these representations in order to extract important information [4]. For example, astrophysics students can learn to quickly ascertaining the general life-stage of a star based on its location on a Hertzsprung-Russell diagram (a graph of luminosity vs temperature).

In this paper, we explore the relationships between the form of an external representation, its function, and its meaning in an analysis grounded in distributed cognition. We consider the questions: How do we encode information and meaning in representations? What role does the medium (paper, whiteboard, computer) play? What characteristics help us describe the types of representations used in physics? How might these characteristics impact the use of representations by novices and experts?

We begin with background and relevant prior work, then present our specific research questions. In our analysis, we argue that external representations can be described as organized information about a conceptual referent presented in a medium. We then describe more detailed characteristics, which we term structural features, that we propose provide insight into student difficulties with representations and help make instructional decisions. We present several examples to illustrate these ideas before discussing implications for pedagogy and future research.

II. BACKGROUND AND RELATED WORK

Our analysis draws on ideas and language about cognition from several theoretical perspectives including, most prominently, distributed cognition. We present some relevant background from those fields and implications for instruction, then discuss related work in physics education research.

A. Knowledge Structures

Cognitive scientists identify two types of memory, working memory and long term memory. Working memory has a limited capacity (7 ± 2 items) and short duration (a few minutes) [5–6]. Long term memory has a greater capacity and information can be retained for many years. One can increase the information capacity of working memory with strong links between information to form a “chunk,” a strongly associated set of information that is accessed as one item that can be unpacked as needed.

The idea of chunking is an example of how knowledge can be structured. In cognitive science, this structure is
referred to as schema theory, in which nodes or elements are linked (the strength of these links can vary) and activated. The more often elements are activated together, the more strongly linked they become. The information activated in a given situation depends on the individual’s knowledge structure and the situation’s context. Sophisticated, flexible reasoning requires a highly interconnected knowledge structure. The knowledge of more experienced learners, such as expert physicists, is more interconnected and organized in ways that are productive for physics reasoning than less experienced physics learners. When a piece of information is activated (brought into working memory), other information that is strongly linked is more likely to be activated than information that is more weakly linked. A new piece of information that is strongly linked to elements already in long term memory will be easier to access again in the future.

Research on knowledge structures and problem solving often conceptualizes people as novices or experts to capture differences in knowledge and reasoning. Novices are usually described as having less-interconnected knowledge structures while experts have highly-interconnected knowledge structures, which leads experts to hold more information in their working memory.

We expect that these differences in knowledge structures impact how a person interacts with external representations. A person having a more-interconnected knowledge structure in a domain, as professionals often do in their field, will be able to understand abstract or information-rich representations more quickly, make connections between representations more easily, and may even be able to perform more tasks “in their head.” In our analysis, we consider how differences in knowledge structures, particularly working memory capacity and chunking, might interact with features of representations during tasks.

B. External representations

Representations can be either external (to the mind — such as a graph displayed on a computer screen or equations written on a sheet of paper) or internal (to the mind — such as a graph or an equation in your head). In distributed perspectives of cognition, external representations are considered a part of the cognitive system that support cognition in many ways \[7,9\]. They store information, extending both working and long-term memory. They can be reformulated to a new type of external representation (e.g., drawing an equation from a graph). They can be spatially rearranged so that new relationships can be perceived (e.g., algebraic manipulation to solve for an unknown). They must be constructed incrementally and serially, allowing constraints to be considered and resolved \[9\]. Internal and external representations transform each other. For example, when someone sees an external representation, like an equation, they can internalize it — they might see and manipulate it in their mind. They can then modify the external representation to mirror the internal representation and perceive new relationships from the external representation.

Distributed perspectives of cognition treat external representations as tools that aid in performing tasks. Considering the task is important for understanding how the tool works and for identifying what features of the tool are important. Using external representations allows a person to solve more complicated problems by offloading some of the cognitive burden onto the representation. Sometimes a person wants to perform a task that exceeds the capacity of their working memory and they use external artifacts or tools to perform these tasks. For example, most people write down algebraic manipulations if doing more than a few steps.

While using external representations can be very powerful, it can also be chunky and slow. Becoming an expert involves building a rich knowledge structure with many internal representations, which is built by having extensive experience with external representations. Learners can become more efficient by internalizing representations as their knowledge structures become more interconnected, while chunking increases their effective working memory.

External representations are also means of communicating ideas — they are sharable objects of thought \[9\]. This role of external representations is central in instruction. Instructors communicate information to learners by talking, writing, drawing, gesturing, etc. — all external representations of ideas. Learners must interpret these representations. Learners also communicate their level of understanding to the instructor with external representations and instructors interpret these to assess students formally and informally.

External representations have conceptual and material features \[10\]. Conceptual features are the ideas that the external representation represents. The material features are a consequence of the physicality of the representation — the materials it is constructed with, its specific shape/color/size, etc. Just as the structure and properties of our memory affect how we think, the structure and properties of external representations (their material and conceptual features) affect how they are used as part of a cognitive system \[5\]. For example, a function of two variables (like the temperature of a floor) can be represented externally in several ways: as an equation, a table of data, a 3-dimensional surface plot with the height of the surface representing the value of the function, or a contour map. All of these representations represent the same function but in different ways. These differences facilitate or constrain certain ways of thinking and how operations are performed. For example, a table of data is necessarily discrete while the other representations are continuous. Local extrema are easy to perceive with the surface but finding them with an equation requires setting derivatives to zero. Determining a derivative with an equation can be done by manipulating symbols using algebraic rules, while determining derivatives from a
table of data or a contour map requires taking ratios of changes.

In this paper, we largely consider tasks that are performed in the contexts of learning physics and solving physics problems. Therefore, rather than conceptualize people as “novices” or “experts”, as is common in research about knowledge structures and problem-solving performance, we find it useful to talk about “learners” and “professionals”. This language reflects differences in how external representations might be used as well as differences in disciplinary knowledge.

C. External representations in physics and mathematics learning

Representations shape, and are shaped by, disciplinary practices. The specialized representations used in physics arose from, and have evolved over time in response to, a disciplinary need [11]. In other words, the function of these representations and their form influence each other. As a result, specialized disciplinary representations may be complex, “non-intuitive” and therefore present challenges for learners.

For example, Elby noted that learners tend to be more successful in interpreting graphs which look like what they represent — a “what you see is what you get” or “WYSIWYG” characteristic [12]. A plot of the trajectory of a projectile (height vs. horizontal distance along the ground) looks like the projectile’s path in the air. This plot of trajectory is much easier to interpret than a more abstract plot of, say, the horizontal component of speed vs distance along the ground, which bears little resemblance to the path of the motion. Elby characterized WYSIWYG as a fine-grained intuitive knowledge element whose activation depends on context. His analysis draws attention to the ways that a representation’s form and structure can influence its interpretation and use, and thereby pose a challenge for physics learners.

To explain learner difficulties with interpreting abstract representations in physics, Podolefsky & Finkelstein [13] developed a model of analogical reasoning about representations. This Analogical Scaffolding model suggests that students can build knowledge structures for interpreting an abstract representation with the help of instruction that appropriately sequences concrete (i.e., more WYSIWYG) and abstract representations. In their empirical work, they showed that such an instructional sequence led to more success in solving related problems than instruction that included only concrete or abstract representations. We will argue that the level of abstraction of a representation is just one of many structural features of representations that affect learning (in our analysis below, we call this structural feature literalness). Additionally, by considering other structural features of representations, we can suggest other potentially beneficial instructional sequences.

Previously, we examined structural features of symbolic quantum notational systems [14]. Structural features are a blend of conceptual and material features that influence how symbolic quantum notations are used for making calculations. Our structural features of quantum notation systems include: externalization, compactness, individuation, and support for computation. These features are centered on the representation with some attention to how the representation is used. Wan, Emigh, & Shafer [15] use this framework to argue that the structural features of wavefunction and Dirac notation impact student reasoning about probability amplitudes, probability densities, and probabilities. However, our analysis and Wan, Emigh, & Shafer’s work were limited in both topic (quantum mechanics) and representational type (algebraic). Given the wide variety of representations used in physics, we were motivated to perform the more comprehensive analysis presented in the current paper. Here, we consider a much larger set of representations beyond symbolic quantum notations and therefore propose a much more expanded set of structural features of external representations used in physics and physics learning.

Shermerhorn, et al. [16] adapted the structural features framework into a set of computational features to understand students’ approaches in calculating quantum expectation values: either matrix multiplication, integration, and summation. These computational features are: identification (the extent each of the elements of the expectation value expression are identifiable to the student), externalization (expression of elements and features of the solution method for the physical situation), computational confidence (a measure of the student’s comfort with a particular calculation), and computational compactness (space taken for writing and/or calculation). Shermerhorn, et al., use these characteristics to address questions of student difficulties and preferences. Their work illustrates how an analysis centered on the external representations can provide a productive foundation for studies of student learning and problem solving.

A main difference between the aforementioned structural features and these computational features is that while the structural features are centered on a representation, Shermerhorn, et al.’s, computational features describe the student and the student’s calculation. “Confidence” is an attribute of the student with respect to a particular computational approach. “Identification” is an attribute of the students’ knowledge of the elements of a computational approach. By encompassing attributes of the student, computational features can be used to evaluate an individual student’s problem-solving performance. In this paper, as in our previous work, the structural features are centered on the external representations and do not include student attributes. For example, when we consider the amount of specialized knowledge needed to use a representation, we center this structural feature on the representation (e.g., “Mathematica requires knowledge of specialized syntax” as op-
posed to “This student uses Mathematica syntax”). Also, although we do not aim to evaluate the performance of users, we consider the implications of the structural features on performance when users are in “learning” and “professional” situations.

In her framework for designing instruction with multiple representations, Ainsworth [17] similarly identified a set of characteristics of representations that influence the difficulty of relating representations. She suggests that “the more the formats of the representations and the operators that act upon them differ, the more difficult it will be for learners to translate between them.” This set of characteristics includes: the sensory channel of the representations (i.e., auditory vs. visual), the modality of representations (i.e., text vs. diagram), the level of abstraction, the specificity (the extent to which the representation permit abstraction), the type of representation (e.g., histogram, equation, table, line graph, narrative text, picture), the level of integration (i.e., integrated presentation vs. presented separately), whether representations are static or dynamic, and the dimensionality (2D vs. 3D).

The set of characteristics proposed are different from Ainsworth’s set in several ways. First, the purpose of the two sets are different: Ainsworth’s characteristics address relating multiple representations while our structural features try to characterize using individual representations. Therefore, the integrated presentation characteristic does not appear in our set. Second, we seek to distinguish more subtle differences. For example, Ainsworth’s characteristics do not help us understand the differences among symbolic/algebraic quantum notations; all of Ainsworth characteristics are the same for equations written in Dirac bra-ket notation, matrix notation, and wavefunction notation. Yet, learners perform quite differently with these different notations. Similarly, we would like to characterize differences in situations like solving a physics problem using pen and paper versus with a computer (different media), differences which are not easily captured using Ainsworth’s characteristics.

III. THEORETICAL PERSPECTIVE AND RESEARCH QUESTIONS

The cognitive science and distributed cognition perspectives described above guide our research questions and analysis. Given cognition science’s focus on how people process information, we attend to how memory and the organization of knowledge affect doing and learning physics. The focus on distinctions between experts and novices can highlight the growth in information processing perspective, we attend to how using and interpreting a representation’s conceptual features depends on an individual’s knowledge structure. Expert-novice differences in information processing lead us to expect implications for learning physics.

In this theoretical analysis, we investigate:

RQ1: What aspects make one instance of a representation different from another?

RQ2: What higher-level structural features of external representations arise from the interplay between their material and conceptual features?

RQ3: How do we expect these aspects and emergent structural features to impact how people with different levels of experience (learners and professionals) use and make sense of external representations?

RQ4: What are the implications of these aspects and emergent structural features for teaching and learning?

In addressing these questions, we propose a framework for describing representations, and illustrate this framework with examples of different representations used in physics. We begin by describing three aspects of representations that we argue allow one to distinguish one representation from another. We then present a set of nine structural features that can be used to describe representations with greater specificity. These nine structural features include the four we introduced previously [14]. While this theoretical analysis is informed by our experience working with students as instructors and physics education researchers, we do not present data as such in this paper. We nevertheless believe this analysis to be a useful contribution that will inform future research efforts (see section V).

IV. ANALYSIS

A. Three Aspects of Representations: Organization, Medium, and Referent

First, we face a basic question: what makes one instance of a representation different from another? By ‘instances’ of a representation, we mean a specific example
of a representation: the graph drawn by hand on a whiteboard is a specific instance, and the graph generated by a computer is another. It seems obvious that different types of representations are used in physics, and that we would recognize differences among graphs, diagrams, algebra, and models (for example). But are the differences between a graph drawn by hand on a whiteboard and one generated by a computer important? What about the differences between a force diagram and a circuit diagram?

In thinking about external representations, and in distinguishing different representational instances, we find it useful to consider three aspects of a representation:

- **Organization**: The way information is organized in space (or time), such as representing displacement on the vertical and time on the horizontal in a graph.
- **Medium**: The material in which the representation is created, such as paper, whiteboard, computer, or even sound. Physical models may be created out of plastic or other materials. (Perhaps the mind could be considered a medium for internal representations, but this is a very special case that we do not explore further here.)
- **Conceptual Referent**: The thing that the representation is representing, such as a function or physical system.

Considering these three aspects, we can describe an external representation as **organized information** about a **conceptual referent** presented in a **medium**. The way information is organized in space uses disciplinary knowledge and conventions to give meaning to the representation of the thing (referent) without actually being that thing. The medium ‘holds’ the representation, provides constraints and affordances, and mediates our interaction with the representation. Consideration of these three aspects guided our identification of the structural features of representations presented below; we discuss the aspects in some detail here before discussing structural features.

1. **Organization of information**

Physicists employ a great variety of representations that are, in a literal sense, marks on paper. The way information is organized in space (which necessarily implies the meaning and interpretation of that organization) is an important part of differentiating one type of representation from another [24]. Consider, for example, two recognizably different representations: a topographical contour map and a graph of elevation as a function of distance along a path (an elevation profile). Both contain some information about elevation along a path, but the information is organized differently. An equation for elevation as a function of location would be yet another representation; the information, which only has meaning inasmuch as one can interpret the symbols, is organized all together differently.

In an external representation, physical space often represents some other quantity. For example, in a graph of altitude as a function of time for a falling object, vertical distance on the graph represents vertical distance in physical space, while horizontal distance on the graph represents a non-spatial quantity (time). Using physical distance to represent time or another non-spatial quantity is so ubiquitous in physics that it may take some effort for experts to recognize this usage as a significant abstraction.

Physicists often go further and use the same spatial direction to represent two or more non-spatial quantities, such as an energy diagram in quantum mechanics (which is actually a nested graph [15] that shows on the vertical axis both wavefunction probability density and energy. Similarly, a force diagram uses space on the paper (or whiteboard, etc) both to represent force magnitude (length of the arrow) and location in space (direction of the force and sometimes a drawing of the objects). In contrast, a circuit diagram uses space on the diagram to represent the topology or nodes of the actual circuit, but the lengths, positions, and paths of the connecting wires are only suggestive.

2. **Medium**

Two representations with the same spatial organization may be rendered in different media. For example, a graph of projectile motion may be rendered on a dry-erasable whiteboard or on a computer screen. Both might organize information in the same way, for instance by using space in one direction to represent vertical position and space in an orthogonal direction to represent time, with a curve indicating the object’s position at the corresponding time. However, we will likely interact differently with a graph on a whiteboard than with one on a computer — the medium matters. Using a whiteboard is relatively straightforward and intuitive. Whiteboards are easy to revise and, especially with larger boards, provide a shared workspace that facilitates collaboration. Most of us have little reservation about drawing on a whiteboard, even if we are unsure of the physics. Creating a graph on a computer, in contrast, requires some specialized knowledge, such as the technical knowledge of how to use the computer and the commands and syntax for a program like Mathematica. Furthermore, and perhaps more fundamentally, to create such a graph using software like Mathematica, one needs an algebraic representation of the curve but no knowledge of the shape [25]. This may be useful or a hindrance, depending on one’s experience and knowledge of the problem. If one derives an equation for the motion of a falling object experiencing drag or other complications, using a computer might be the easiest way to visualize the motion, though one can instead generate a curve from a formula ‘by hand’.
The process of changing, saving, and sharing the graph is different on a whiteboards than a computer as well. Whiteboards, paper and pencil, and printed materials are common and similar media; compared to a computer, their differences are slight. However, most of us are more willing to ‘doodle’ on a whiteboard than with pen and paper. External representations are persistent in time, which is one of their great advantages [19]. Some media, such as whiteboards, are more ephemeral. This can be an advantage that invites brainstorming, but is a drawback if one wants to save or share the work. Whiteboards are generally better as shared spaces for collaboration, but are not as portable or permanent as work on paper. Taking a photo of a whiteboard is a common workaround but are not as portable or permanent as work on paper. A printed graph may have the accuracy to support measurement and calculation, such as solving a transcendental, which is typically lacking on a whiteboard sketch.

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3. Conceptual referent

Different representations, with different spatial organization and/or media, might be used to represent the same referent. For example, in physics we might represent an electric field with vector arrow maps, a graph of the field magnitude as a function of time, an algebraic expression, a field line diagram, or a table of numerical values. The electric field is the conceptual referent for all these different representations. Each representation captures aspects of the electric field, but no one representation captures all aspects of a referent. Different representations may highlight different aspects of the referent. In the electric field example, field lines highlight sources and flux, a vector field map can give a visual impression of the shape and symmetry of the field, and a graph versus time emphasizes the time dependence. Depending on which aspects of a concept are of interest, and the knowledge of the user, some representations may be more useful or relevant. For example, if one wishes to draw equipotential curves, starting with either a vector arrow map or a field line diagram would be very useful because equipotential lines could be drawn directly on, and in coordination with, the diagram. An algebraic expression for the field could also be used to produce equipotential curves, but this process would be more difficult than either choice above, even for an expert. A graph of field magnitude versus time would be of little help for this task.

B. Structural features of representations

The three aspects discussed above can help focus and organize our thinking about external representations. They provide a way to understand how the various external representations used in physics are different from each other. But these three high level aspects do not provide sufficient detail for understanding student difficulties with using external representations, or making decisions about the pedagogical use of external representations in teaching physics. In this section, we describe structural features of external representations and outline some implications for novices and experts. Most of these structural features link one or more of the aspects described above. For instance, fidelity to referent / literalness, which describes how literally the representation captures the thing being represented, concerns the relationship between the spatial organization of information and the conceptual referent. (To a degree, this also concerns the medium, in as much as the medium structures and constrains the organization of information.) As a second example, configurability (how easily the representation can accommodate changes in parameters) depends on medium and organization of information. An equation on a whiteboard is more configurable than a printed table of data.

The present analysis expands on our work in a previous paper [14]. There, we described four structural features of algebraic representations of quantum mechanical states: compactness, externalization, symbolic support for computation, and explicitness were used to characterize matrix, wavefunction, and Dirac notation. This approach provided insight on the different affordances and limitations of the representations. These different representations (matrix, wavefunction, and Dirac notation) organize information differently, and in that study, the conceptual referents and the medium were the same (quantum states for a spin system and for an infinite square well written on a whiteboard). Here we greatly expand that analysis by describing nine structural features (including the four presented in our earlier work) relevant to how people use and learn representations. While each of these captures a distinct feature of external representations, there may be some overlap and we do not claim that these are completely exclusive or inclusive. Table I and II lists and describes the structural features, along with examples and implications for novices and experts. We have decided not to define the structural features in the main text but rather in tables. We argue that this presentation is reasonable and efficient, as it leverages the structural features of the table representation: compactness in avoiding redundancy with the text, individualization in separating the structural features listed in the table, and support for procedures in helping readers find specific entries quickly. In the following section, we illustrate these structural features further through examples where we make detailed contrasts between different representations.
<table>
<thead>
<tr>
<th>Structural feature</th>
<th>Description</th>
<th>Examples</th>
<th>Implications for learners and professionals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fidelity to referent / literalness</td>
<td>How literally the representation captures the thing being represented.</td>
<td>A graph of trajectory (y vs x) has high fidelity: it &quot;looks like&quot; the thing it represents; Montessori counting beads (high fidelity) vs roman numerals (lower fidelity) are different ways to represent numbers; A 3D surface can represent a hill (high fidelity) or an electric potential (low fidelity). Fidelity differs for different referents; representing sound with a diagram of air molecules is more literal than a graph of displacement versus location.</td>
<td>Novices may find more literal representations easier to interpret, while more abstract representations are more difficult. This structural feature is central to analogical scaffolding and Elby's WYSIWYG element. More knowledge of the discipline is required to interpret more abstract representations.</td>
</tr>
<tr>
<td>Externalization</td>
<td>The degree to which elements and features are externalized with markings included in the representation.</td>
<td>A verbally rich description has greater externalization than a label or term; graphs without labels have lower externalization than those with; in quantum mechanics, wavefunctions have greater externalization than Dirac notation.</td>
<td>High externalization helps store or offload information (helpful for reducing cognitive load, especially for novices, who typically chunk much less than experts). Externalization is more useful and important if one has fewer internalized representations to connect with or a less well developed and connected knowledge structure.</td>
</tr>
<tr>
<td>Compactness</td>
<td>How much space and writing the representation requires.</td>
<td>A table of data is less compact than a graph; graphs tend to be more compact than physical models; in quantum mechanics, Dirac notation is more compact than wavefunctions.</td>
<td>More compact representations are easier to manipulate and support chunking better.</td>
</tr>
<tr>
<td>Individuation</td>
<td>The degree to which important features are represented as separate and elemental.</td>
<td>In Dirac notation, quantum basis states are written as kets that have a high degree of individuation compared to wavefunctions, where coefficients, normalization constants, and basis states may be combined algebraically.</td>
<td>Higher individuation may help novices map conceptual features into an external representation, while lower individuation may obscure the distinct meaning of separate elements. Novices may not know how much or what detail to include (the criteria for deciding may be unknown as well). Experts may choose to overlook or ignore detail as needed and desired, but notices may be overwhelmed by detail.</td>
</tr>
<tr>
<td>Precision</td>
<td>The amount of detail and accuracy included in the representation and available for readout.</td>
<td>A sketch of a graph only qualitatively conveys relationships and has low detail, but a carefully constructed graph can be very detailed. Mathematica can generate (nearly) arbitrarily accurate graphs. It is difficult to read out values from a perspective drawing of a surface (low accuracy/detail), but a family of curves in 2D (based on different values from the third dimension) has more accuracy and detail. An algebraic expression like f' has less detail than df/dx.</td>
<td>High configurability may be useful for experts, but the knowledge required for reconfiguration depends on the representation. Some constraints on configurability might be productive, especially for novices.</td>
</tr>
<tr>
<td>Configurability</td>
<td>How easily the representation can accommodate changes in parameters.</td>
<td>The period of a sine function in mathematica is more easily configurable than a hand drawn graph if one knows the syntax; A slider bar on a mathematica notebook makes it more configurable but constrains which parameters can be changed. A 3D surface is not configurable, but can be annotated (drawn on). Printed graphs can be annotated but not altered. Graphs on whiteboards can be easily altered.</td>
<td></td>
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</table>

TABLE I: Structural features of external representations, part 1.
<table>
<thead>
<tr>
<th>Structural feature</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Tolerance to noise</td>
<td>How useful or recognizable the representation is when created “sloppily”.</td>
<td>The algebraic representations $x^2$ and $x^4$ are very tolerant to noise (you can still tell them apart even if poorly handwritten), but graphs of these functions have low tolerance to noise and may be easy to confuse; musical notes on a scale have a greater tolerance to noise than notes played out loud; algebraic representations of hermite polynomials versus graphs (quantum harmonic oscillators) — while individual characters in formula are tolerant to noise, the entire formula is sensitive to getting every term correct; the graphs are different enough that they can be distinguished even if not completely accurate.</td>
<td>Novices may not know which conceptually relevant or essential features must be preserved in a representation, and which may be ignored.</td>
</tr>
<tr>
<td>Support for procedures/computation</td>
<td>How the shapes of the symbols or structure of materials support the procedures needed to use the representation for computation.</td>
<td>The shape of angle brackets in Dirac notation can support calculations of expectation values, normalization (as in earlier paper); a determinant or cyclic diagram can help structure a calculation of a cross product; a force diagram can help support choosing axes and decomposing vectors into components (in some cases, like statics, a force diagram may support calculation of a single missing force via geometric vector addition); graphs can support the solution of transcendental equations that can’t be solved algebraically; place value and base ten number system support arithmetic calculation.</td>
<td>Additional support for computation may help guide novices, reduce cognitive load, and reduce computational errors. Experts probably only benefit from such support in more complex or unfamiliar situations.</td>
</tr>
<tr>
<td>Ease of use (of generating and working with the representation)</td>
<td>How much specialized knowledge is required to use the tool that generates or interacts with the representation (not conceptual knowledge of the referent or subject matter).</td>
<td>Pen and paper has high ease of use compared to a computer algebra system which has specialized syntax; using an inclinometer to measure slope on a surface has low ease of use.</td>
<td>Novices, or experts lacking knowledge of the tool, will not be able to use it effectively.</td>
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</table>

**TABLE II: Structural features of external representations, part 2.**

### C. Examples

1. **Force diagram and equations**

First, consider mechanics a problem where two forces act on an object (see Figure 1). Representations of this situation would commonly include both a force diagram and an algebraic equation (Newton’s Second Law). Comparing those two representations, the diagram has higher fidelity to the physical situation than the equation. The diagram evokes the physical objects and geometric relationships (making the magnitudes visible and coordinating them with locations of objects and directions in space). The directions of the vectors are externalized with arrows in the force diagram. Contrastingly, in the equation the direction of the forces is encoded in the
or interpret a result. The compactness can help students write a correct equation, catch errors, and textbooks recommend that students translate the expression \[21\]. The value of this approach can be un-
tenuously using that representation to generate an algebraic problem statement by constructing a force diagram and and externalize the force diagram, which does not lead to confusion or mistakes such as incorrectly resolving a force into components (common on inclined plane problems). Both representations are configurable, though to different degrees. Changing the mass, for instance, requires changing the length of the arrow representing the weight force. In the algebraic equation, the symbol \( m \) would not need to change.

Now we explore how these structural features might affect an attempt to solve a typical problem: given an object’s mass and upward acceleration, determine the tension in the rope pulling the object up. Instructors and textbooks recommend that students translate the problem statement by constructing a force diagram and then using that representation to generate an algebraic expression \[21\]. The value of this approach can be understood in terms of the structural features described above. A force diagram, with higher fidelity to the physical situation, will be easier to generate from the problem statement than an algebraic expression. This fidelity facilitates identifying the forces and relative directions of forces and acceleration. In setting up or solving an equation, the lack of externalization in the algebraic representation (of the vector direction, as discussed above) means it is easier to make a sign error. Coordination with the force diagram, which does externalize the directions, can help students write a correct equation, catch errors, or interpret a result. The compactness and tolerance to noise of an algebraic equation are efficient for solving for the tension.

2. Surface Model vs. Contour Map

As another example, we compare and contrast a dry-erasable contour map of a function of two variables and a 3-dimensional plastic surface model of the same function. Examples of both representations are shown in Figure 2. If the function that is represented is the height of a hill (with latitude and longitude as the two independent variables), the surface model is quite literal; it looks very similar to the thing it represents. A contour/topographic map, with curves representing constant height, is less literal. One needs some specialized knowledge (ease of use) about how to interpret the curves on a contour map, particularly the convention that the adjacent contours represent equal changes in the value of the function. The qualitative behavior of the function is easier to interpret on the surface model: when the height of the surface model is small, the value of the function is small. In contrast, on the contour map, the values of the contours have to be read and interpreted. Moreover, the surface spatially separates or individuates the values of the independent variables in the domain from the value of the function (as height), while a location on the contour map has the two coordinates in the domain as well as a value of the function (these different quantities are not spatially separated). The 2-dimensional nature of the contour map more strongly emphasizes the 2-dimensional nature of the domain. Identifying extrema on a surface model is easy. Identifying extrema on a contour map, particularly distinguishing maxima and minima, requires more interpretation. If the surface model does not have any numbers displayed (not externalized), a measurement must be made to quantify the value of the function. These measurements have limited precision. However, all of the values of the function in the domain are visible (i.e., the surface model exists everywhere in the domain). In contrast, the contours on a contour map are often labeled (externalized) but the values of the function between the contours are not explicitly represented and have to be extrapolated.

A common computation for such representations is to determine the derivative of the function. The surface model and the contour map support this calculation very differently. The derivative of the function is represented on the surface as the slope, which strongly suggests interpreting the derivative as an angle. The slope is a local quantity that describes a small region on the surface.

In contrast, the contour map strongly suggests thinking about the derivative as a ratio of changes (or as the concentration of contour lines). The relevant changes are often much less localized than slope on a surface. An inclinometer (a tool that can be used to measure the slope of a surface) bridges the notion of derivative as a slope to the derivative as a ratio of changes. It "blows up" the...
slope of the surface from a very local property to be one that is extended in space so that a ratio of rise over run can be calculated.

3. **Representations of a Quantum Particle Confined to a Ring**

A typical way to represent states in quantum mechanics is by graphing the probability density (the squared norm of the wavefunction) as a function of position. As an example, consider the probability density of a superposition state for a quantum particle confined to a ring (Figure 3). In this graph, the particle is most likely to be found in regions of high probability density (near $\phi = 0$, $\pi$, or $2\pi$) and least likely to be found in regions with low probability density, (near $\pi/2$ and $3\pi/2$). The features of this plot can easily lead to misinterpretation. First, plotting the angle $\phi$ on the horizontal axis disguises the fact that $\phi = 0$ and $\phi = 2\pi$ in fact are colocated on the ring. They are the same place! This lack of literalness can be addressed by wrapping the $\phi$ axis into a ring, as shown in Figure 3.

Second, using the vertical axis to represent the value of the probability density leads some students to believe that the particle oscillates in this vertical direction (e.g., off the ring). Using a spatial axis to represent the value of the probability density distinguishes or individuates the value of the probability density from the location on the ring - but it allows students to conflate the probability density with positions above the ring. This conflation can be addressed by using color to represent the value of the probability density (Figure 3).

The fidelity of the color-scale ring to the physical system (literalness) supports a correct interpretation of the probability density distribution. However, the color-scale ring lacks detail and accuracy, making it less useful in professional contexts. One cannot easily read off values of probability density for specific values of $\phi$, even with the aid of a color scale (precision). Also, the periodic behavior is evident but sinusoidal behavior cannot be easily distinguished from other periodic functions (such as a sawtooth function).

Given these considerations, an instructional sequence might start with a color-scale ring, as in Figure 3. This is the most literal to the physical situation, geometrically, and can help build conceptual understanding. Then the ring can be presented with probability density plotted on a vertical axis, as in Figure 3, followed by discussion of the canonical graph, Figure 3a. For an example, see ref. “Particle Confined to a Ring” activity and McIntyre, Section 7.5, pp. 218-227. This Analogical Scaffolding approach could then transition to equations and graphs of probability density versus angle for problem solving applications.

4. **Solving equations on paper versus the computer**

Next we consider the process of solving a problem that involves algebra and calculus. In this example, we focus on the role of different media for the representations — namely, paper and computer — while primarily using algebraic representations of the same referent [26]. Here we use ‘computer’ to refer to using a computer algebra system, such as Mathematica, Matlab, or Maple. This analysis is similar to that of Hutchins, who describes differences in doing Distance=Rate*Time problems ‘by hand’, with a calculator, using a nautical slide rule, or using a navigational rule of thumb [7, p. 147 in]. He concludes that “each tool presents the task to the user as a different sort of cognitive problem requiring a different set of cognitive abilities or a different organization of the same set of abilities” [7, p. 154]. We will see that this description also applies to the computer versus paper comparison presented here.

To be concrete, consider problem 2.26 in Griffiths’ *Introduction to Electrodynamics* [22], which asks students to calculate the electrostatic potential due to a charged cone. Setting up the integral is difficult for most students, and involves both physical and mathematical reasoning. The computer is no help with this aspect of the problem. Indeed, the brainstorming, sketching of diagrams, and scratch work needed at this stage is easiest to accomplish on paper (ease of use). Further, the type of small algebraic simplifications and rearrangements required when setting up the problem can be easier on paper. Making algebraic simplifications while simultaneously composing the result with the correct syntax is demanding, especially for students who are still learning to use a symbolic algebra application. Furthermore, the computer may not arrange or group terms in a physically meaningful way. However, in calculating the potential due to a charged cone, one eventually needs to solve a difficult integral, a largely mathematical calculation that not does not involve physical reasoning. The computer can help, or, essentially, do, that part of the problem without the user knowing integration techniques and without making algebraic or other mistakes in the calculation (precision). In contrast, one must know significant amount of algebra and calculus to perform the calculation on paper.
FIG. 3: Three representations of the probability density of a superposition state for a quantum particle confined to a ring: (a) a graph of probability density versus position on a linear axis; (b) graph of probability density, including color-scale, with position wrapped into a ring; (c) probability density represented only by color-scale on a ring.

Although using the computer requires less knowledge of integration techniques, one must know the syntax and commands to use the computer applications (again, relating to ease of use). In this regard, however, the program can present help menus, suggest operations depending on context, and link to other resources, thereby providing support for procedures/computation. Regarding syntax, the computer system has lower tolerance to noise (in this case, syntactic errors, mismatched delimiters, etc.) [27].

The ability to copy and paste on the computer can make repeated or highly similar things easy to generate, which can mitigate issues with lack of compactness in long algebraic expressions. However, the output from computer algebra systems often does not group terms in physically meaningful ways that facilitate interpretation or chunking. The computer may not incorporate the user’s tacit assumptions that a variable or solution is real, bounded, etc., and therefore give results in several conditional cases or without simplifying. In contrast, when working on paper one typically places terms with constants at the beginning of an expression and groups terms with similar powers of a variable, for example. These practices are informed by experience and convention [28].

When working on paper, progress may be limited by one’s knowledge (experience and memory) of mathematical tools and tricks, such as integration techniques or algebra moves. If one is determined to solve the problem, he or she may consult textbooks, talk to others, or search online (using a computer!) for help, possibly learning new techniques in the process. In contrast, the computer handles the calculation in a way that is largely invisible to the user and does not require learning how the solution was obtained. However, tutorials may show a step-by-step solution, allowing one to learn the method if interested. A computer can facilitate ‘exploring’, i.e., trying out different approaches, looking up and pulling in other resources, following leads suggested by the computer, in a way that can be slow, unlikely, or impossible when working on paper and pulling books off the shelf. Nevertheless, treating the computer as a ‘black box’ can be a problem if physically significant choices still need to be made (i.e., which quantities are less than one and therefore can be expanded as a converging power series). Conversely, some steps of solving a problem on paper may benefit from physical insight, which can be used to guide or even transform the calculation (i.e., recognizing that an integrand is odd and therefore the integral is zero), something experts often do. If instructors want to promote this kind of reasoning by students, it is important to recognize this difference between using paper versus a computer. Other aspects of solving a problem are rote and uninteresting (from a physical perspective), and little may be lost in having the computer do them.

Working strictly on either paper or the computer is artificial. If, as instructors, we ask students to eschew using a computer (as on exams), we should have a rationale based on pedagogical goals and an understanding of the representations. More realistically, one would coordinate the two representations, and instructors should help students learn to do this coordination effectively, recognizing that it will be more productive than using either approach alone. For instance, a computer can help yield physical insight, such as by graphing a function, that would be harder to achieve on paper. This example draws our attention to the questions of how these two media work together and support each other and how to use them together effectively. The answers are likely to be different for learners and professionals.

V. PEDAGOGICAL IMPLICATIONS AND RECOMMENDATIONS

The analysis of external representations presented above has several implications for instruction.
A. Instructional Considerations

1. Differences between Learners and Professionals

Representations that are helpful to learners may seem clunky to professionals. Uninitiated learners benefit from more literal representations that externalize important information and do not require specialized knowledge to interpret. Professionals benefit most from compact and configurable representations; these structural features leverage the professional’s extensive and interconnected knowledge structure. For example, while professionals typically use contour maps to represent functions of two variables, students may benefit from starting with a plastic surface model, which may be more literal, represent every value of the function, and allow for quick identification of the qualitative nature of the function with little specialized knowledge. Another way to think about this transition is in terms of information density. Learners may struggle to use representations with high information density: high (or absent) externalization, low individuation, in a compact, highly configurable, and highly detailed form.

2. Considering Types of Reasoning

Representations with different structural features may structure cognitive demands differently, and instructors should attend to what thinking they want students to do themselves versus what thinking is offloaded to an external representation. This distinction is probably most obvious when comparing doing a calculation by hand with pencil and paper versus using a computer application (e.g., having students practice integration techniques or use Mathematica to compute an integral). Ultimately, we hope students are able to use all representations that are relevant for a particular problem, and select the ones that are most productive. But, for instructional purposes, we may wish to guide students so that they develop particular skills or experience. For instance, in a given situation: Do you want your students to find the extrema of a function using algebra (by setting a derivative to zero) or graphically (by visual inspection)? Do you want students to calculate a net force by adding vector arrows or by adding components? Furthermore, instructors should think about how students might coordinate representations in ways that are complementary, as in the case of drawing free-body diagrams to help with writing down algebraic statements of Newton’s Second Law, or expressing an equation in words to better understand its meaning.

3. Orienting Learners

Learners need some orientation in order to understand representations that are more abstract, less externalized, less tolerant to noise, and that require a lot of specialized knowledge. In other words, students need to be taught how a representation depicts the referent and how to relate it to other representations; colloquially, “how a representation is repping what it reps”. This metarepresentational knowledge needs to be explicit in instruction.

B. Instructional Strategies

Based on these implications, we now suggest some strategies for helping students learn to interpret and use external representations. First, we discuss some ways instructors might sequence the introduction of representations to help students become more sophisticated users. Then we discuss supporting students’ metarepresentational competence.

1. Analogue Scaffold to More Abstract Representations

Analogue scaffolding is one such strategy: start with more literal representations and make explicit connections to increasingly more abstract representations as you introduce them. For example, after introducing 3-dimensional surface models to represent multivariable functions, students can be asked to identify with a marker all of the places on the surface with the same value of the function, for several different values. In doing this, students would draw contours on the surface. Students can then be asked to align the surface with a corresponding contour map printed on paper, and explain how they are making this alignment (23). In this instructional task, students must attend to the conceptual connection between the surface model representation and the contour map.

2. Increasing Information Density

Another sequencing strategy is to start with a representation with a lower information density and increase the density over time. One might start with more “digital” representations, as learners may not know the essential features to attend to in a representation. As more “analogue” representations are introduced, the instructor can point out what “noise” can be ignored. One way to do this might be to ask every student to generate a representation on their personal whiteboards, and after which the instructor can compare and contrast the essential features of student-generated representations. Similarly, delaying the introduction of representations with a lot of detail and accuracy may be helpful to learners. Instructors can gradually introduce detail and help students chunk features in order to manage highly detailed representations. Alternatively, instructors can sequence representations in terms of how configurable they are. Highly configurable representations might be introduced
later, when the changeable parameters are relevant to the task. Instructors should take care that each student has access to the configurable representation and has a chance to explore different configurations as part of their learning experience. Less configurable representations may have productive constraints that prevent students from focusing on/exploring parameters that are not relevant to the task at hand. In all of these sequences, we believe it is important to guide students to explicitly make connections between different representations in a sequence.

3. Sequencing from External to Internal

Instructors should try to externalize the internal representations that they use when talking with physics learners. Physics learners are not likely to have the same interconnected knowledge structure as instructors and may not be actively using the same internal representations. Externalizing these representations increases communication between the instructor and the learner (they act as shareable objects of thought) and may help students develop more interconnected knowledge structures.

4. Supporting Metarepresentational Competence

Not only should instructors make the representations they are using external, instructors should explicitly describe how they are using these external representations. Having these metacognitive discussions with students should help students develop knowledge about representations. Additionally, instructors should talk more broadly about the importance and breadth of representations in physics and the advantages of representational fluency in solving physics problems. Learning more than one way of thinking about a problem can be demanding for students, and instructors should validate the effort required. This instructional approach can also provide an alternative to the view that doing physics is simply manipulating the correct algebraic equations, and help students move past giving preference to the first representation they learn. Explicit discussion of the importance of a variety of representations and tasks where students experience for themselves the power of using multiple representations will help students develop a more realistic and productive understanding of doing physics.

Open-ended tasks where students can learn how to both choose and use appropriate representations can support students in developing metarepresentational competence (ref diSessa). In particular, tasks where (1) there is not a single obvious choice of which representation to use, and (2) students are asked to invent their own representation provide opportunities to compare and contrast the affordances of different representations as well as relate or translate between representations.

VI. RESEARCH IMPLICATIONS AND FUTURE DIRECTIONS

The analysis of representations presented here has several implications for future research. We hope that this theoretical analysis provides insight to other researchers and instructors, but naturally one would like to know how these structural features can help interpret student thinking, problem solving, use and interpretation of representations, etc. Future study could also examine if these structural features can productively guide instructional development. For instance, the structural features could be the basis for instructional sequences: from literal to more abstract representations, from more to less noise tolerant, from less configurable to more (removing productive constraints). The structural features could also be used in studying the impact and effectiveness of activities designed to help students develop meta-representational knowledge and understanding.

Several fundamental questions remain unanswered from this analysis. For instance, do the structural features presented here form an exhaustive or exclusive set? Do they group into higher level features? Some of these structural features relate to how abstract a representation is, its information density, or what can be done with it. However, we have not explored these groupings further. The relationship between representations, as described here, and other concepts could be explored as well. For instance, can transfer among representations and representational fluency be understood in terms of changes in the three aspects identified here (medium, conceptual referent, and spatial organization of information)? We suspect that some types of transfer require the ability to move between representations with different conceptual referents (but the same medium and organization of information), while representational fluency draws on understanding representations with the same conceptual referent, but in a different medium and/or different spatial organization of information. Other studies could examine the impact of different media, for example Maple/Matlab versus paper and pencil, or interactive simulations versus physical equipment, and how to effectively design instruction around this. Finally, the relevance of this analysis to fields outside of physics remains an open question.

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[24] Morse code is an example of a type of representation that uses time to organize information, though Morse can also be represented spatially with dots and dashes.
[25] Software like GeoGebra can provide an equation of a plot given the shape in some contexts. Of course, one could sketch a graph on a computer with a stylus or touchscreen, which may be a similar experience to using a whiteboard. Here we are thinking of using graphing software for this task.
[26] throughout this example, whiteboard could be substituted for paper with minor differences.
[27] Some systems, such as WolframAlpha, are more forgiving in this regard, and this ability is likely to improve with advances in computing and artificial intelligence.
[28] These simplifications can be programmed, although not easily.